

KNOT TRACES AND CONCORDANCE

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ABSTRACT. We give a method for constructing many pairs of distinct knots K_0 and K_1 such that the two 4-manifolds obtained by attaching a 2-handle to B^4 along K_i with framing zero are diffeomorphic. We use the d-invariants of Heegaard Floer homology to obstruct the smooth concordance of some of these K_0 and K_1 , thereby disproving a conjecture of Abe in [Abe16]. As a consequence, we obtain a proof that there exist patterns P in solid tori such that $P(K)$ is not always concordant to $P(U)\#K$ and yet whose action on the smooth concordance group is invertible.

1. INTRODUCTION

Conjecture 1.1 (Akbulut-Kirby Conjecture, 1978. Problem 1.19 of [Kir97]). *If K and K' have homeomorphic 0-surgeries, then K and K' are smoothly concordant.*

One might view this conjecture as motivated by the following two ideas. First, the 0-surgery of a knot determines fundamental smooth concordance invariants such as the Tristram-Levine signature, the Alexander polynomial, and the algebraic concordance class of a knot, as well as more involved invariants such as Casson-Gordon signatures, metabelian twisted Alexander polynomials, and those associated to the filtration of [COT03]. Secondly and perhaps more fundamentally, it is a well-known result (see Question 1.19 of [Kir97], [AJOT13]) that, assuming the smooth 4-dimensional Poincaré conjecture, for K and K' having homeomorphic 0-surgeries K is smoothly slice if and only if K' is smoothly slice.

In 1983 Livingston gave examples of knots K which are not topologically concordant to their reverses, as distinguished by the twisted Alexander polynomials of $K\#-K^r$ [Liv83]. Since knot surgeries are insensitive to the orientation of the underlying knot this provided counterexamples to the original Akbulut-Kirby conjecture and led to the following revision.

Conjecture 1.2 (Revised Akbulut-Kirby Conjecture). *If K and K' have $S_0^3(K) \cong S_0^3(K')$ then, up to reversing the orientation of either knot, K and K' are smoothly concordant.*

In 1980 Brakes gave a construction of pairs of knots which share a 0-surgery and yet are distinct as unoriented knots [Bra80]. Using this construction, Gompf-Miyazaki showed the following.

Proposition 1.3. [GM95] *At most one of the slice-ribbon conjecture and Conjecture 1.2 is true.*

However, the next significant progress in the resolution of this conjecture did not come for twenty years, in the work of Yasui.

Theorem 1.4. [Yas15] *There exist infinitely many pairs of knots K and K' with homeomorphic 0-surgeries such that the smooth 0-shake genera of K and K' are different.*

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Recall that for $n \in \mathbb{Z}$ the (*smooth*) n -shake genus of a knot is defined to be the minimal genus of a smoothly embedded surface that generates the second homology of the n -trace $X_n(K)$ of K . (We refer the reader to Section 3.1 for general definitions, for example of the n -trace of a knot.) It is straightforward to verify that n -shake genus is both unchanged by reversal and a concordance invariant, so Yasui's result gives counterexamples to the revised Akbulut-Kirby conjecture. The 0-shake genus is by definition an invariant of the 0-trace, so Yasui's examples certainly do not have diffeomorphic 0-traces. The following conjecture of Abe remained open.

Conjecture 1.5. [Abe16] *If K and K' have diffeomorphic 0-traces then, up to reversing the orientation of either knot, they are smoothly concordant.*

The techniques of annulus twisting (see [Oso06], [AJOT13]) can be used to produce pairs of knots which share a 0-trace. Abe-Tagami consider such a pair and apply a ribbon obstruction due to Miyazaki [Miy94] to show

Proposition 1.6. [AT16a] *At most one of the slice-ribbon conjecture and Conjecture 1.5 is true.*

We extend the techniques of [Bra80] to produce pairs of knots with diffeomorphic 0-traces. We then use the d-invariants of Heegaard Floer homology applied to the double branched cover of our knots in order to disprove Conjecture 1.5 as follows.

Theorem A. *There exist infinitely many pairs of knots (K, K') such that K and K' have diffeomorphic 0-traces and yet are distinct in smooth concordance, even up to reversal of orientation.*

In order to discuss several applications of Theorem A we recall and reprove a result which has been known to the experts for some time [KM78], [GS99].

Theorem 1.7. *K is smoothly slice if and only if $X_0(K)$ smoothly embeds in S^4 .*

Proof. For the only if direction: Consider S^4 and an equatorial S^3 therein, which decomposes S^4 into the union of two 4-balls B_1 and B_2 . Consider K sitting in this S^3 . Since K is smoothly slice, we can find a smoothly embedded disk D_K which K bounds in B_1 . Observe now that $B_2 \cup \nu(D_K) \cong X_0(K)$ is smoothly embedded in S^4 .

For the if direction: Let $F : S^2 \rightarrow X_0(K)$ be a piecewise linear embedding such that the image of F consists of the union of the cone on K with the core of the two handle. Notice that F is smooth away from the cone point p . Let $i : X_0(K) \rightarrow S^4$ be a smooth embedding. Then $(i \circ F)$ is a piecewise linear embedding of S^2 in S^4 , which is smooth away from $i(p)$. Note that $W := S^4 \setminus \nu(i(p)) \cong B^4$ and that the restriction of $(i \circ F)$ to the complement of a small neighborhood of $F^{-1}(p)$ in S^2 is a smooth embedding of D^2 in $W \cong B^4$. Further, if we choose this neighborhood to be the inverse image of a sufficiently small neighborhood of $i(p)$ we have that $(i \circ F)(D^2 \setminus \nu(F^{-1}(p)))$ intersects ∂W in the knot K . \square

An identical proof shows that K is topologically slice if and only if $X_0(K)$ topologically embeds in S^4 .

Corollary 1.8. *Let K and K' be knots with $X_0(K)$ diffeomorphic to $X_0(K')$. Then K is smoothly slice if and only if K' is smoothly slice.*

Using Corollary 1.8 we give a brief proof of the following strengthening of Theorem 3.1 of [AT16b].

Corollary 1.9. *Let J be a knot in S^3 admitting an annulus presentation in the sense of [AT16b] and J_n be the knot obtained from J by the n -fold annulus twist for some $n \in \mathbb{Z}$. Then J is smoothly slice if and only if J_n is smoothly slice.*

Proof. By Theorem 2.8 of [AJOT13], $X_0(J)$ is diffeomorphic to $X_0(J_n)$. \square

Let \sim denote smooth concordance and $\mathcal{C} := \{\text{knots in } S^3\} / \sim$ denote the smooth concordance group of knots. Recall that for a pattern P in a solid torus, the operation of taking satellites by P descends to a well defined map $P : \mathcal{C} \rightarrow \mathcal{C}$.

Definition 1.10. A pattern P is (*smoothly*) *invertible* if there exists a pattern Q such that $P(Q(K)) \sim K \sim Q(P(K))$ for any K .

A particularly uninteresting family of invertible patterns is given by those with geometric winding number one. These *connected sum patterns* act by connected sum even on the monoid of knots up to ambient isotopy in S^3 , and hence certainly descend to invertible maps on \mathcal{C} . An interesting family of winding number one but not geometric winding one patterns consists of *dualizable* patterns, as seen in [Bra80] and [GM95]. Using Theorem A and Corollary 1.8 we give a new proof that many patterns which do not have geometric winding number one are invertible, a result which first appeared in a stronger form [GM95].

Theorem B. Let P be a dualizable pattern, and let $P^{-1} := \overline{P^*}$. Then for any knot $K \subset S^3$,

$$P^{-1}(P(K)) \sim K \sim P(P^{-1}(K)).$$

The work of [DR16] shows that there are dualizable (hence invertible) patterns P whose exteriors are not homology cobordant to the exterior of any connected sum pattern. However, this is not a priori enough to establish that these patterns' actions on the smooth concordance group are distinct from that of connected sum with some knot. However, by combining Theorem B with Theorem A, we obtain the following.

Theorem C. There are invertible patterns which do not act by connected sum on the smooth concordance group.

It is perhaps an interesting question whether there exist invertible patterns P which, despite having exteriors distinct from that of a connected sum pattern, still act by connected sum on the knot concordance group. We also have the following corollary of Theorem C.

Corollary 1.11. *There exist invertible patterns P with $P(U)$ slice such that $P(K)$ is not always smoothly concordant to K .*

We say a pair of knots K_0 and K_1 are (n, m) *0-shake concordant* if there is a smooth properly embedded planar surface in $S^3 \times [0, 1]$ with boundary consisting of n 0-framed copies of K_0 in $S^3 \times \{0\}$ and m 0-framed copies of K_1 in $S^3 \times \{1\}$. Examples of knots which are 0-shake concordant but not concordant first appeared in [CR16]. The pairs we consider in Theorem A give new such examples as follows.

Proposition 1.12. *If $X_0(K_0)$ is diffeomorphic to $X_0(K_1)$ then K_0 is $(1, r)$ 0-shake concordant to K_1 for some $r \in \mathbb{N}$.*

Proof. If K_0 is slice this follows by Theorem 1.8. Otherwise, for $i = 0, 1$ let $X_0^b(K_i) := X_0(K_i) \setminus \nu(\{x_i\})$ for x_i a point in the interior of $X_0(K_i)$. Then $X_0^b(K_0) \cong X_0^b(K_1)$. Consider K_0 in the S^3 boundary component of $X_0^b(K_0)$. This K_0 bounds a smoothly embedded disk in $X_0^b(K_0)$. Therefore the copy of K_0 in the S^3 boundary component of $X_0^b(K_1)$ must bound

a smoothly embedded disk in $X_0^b(K_1)$. Since K_0 is not slice, this disc must intersect the attaching region of the 2-handle of $X_0^b(K_1)$ and the result follows. \square

Let $g_4(K)$ denote the (smooth) 4-genus of K , the minimal genus of any smooth surface that K bounds in B^4 . We observe that the annulus twisting construction as used in [AJOT13], [AT16a] and [AT16b] produces pairs of knots with diffeomorphic 0-traces which each have 4-genus at most 1. By Corollary 1.8, any such pair of knots must have the same 4-genera. In contrast, our techniques can be used to construct pairs of knots with $X_0(K) \cong X_0(K')$ such that $g_4(K)$ is arbitrarily large. Thus it remains possible that there is some such pair with $g_4(K) \neq g_4(K')$. Such an example would give a negative answer to the following question.

Question 1.13. (Question 1.41a of [Kir97]) Must the 4-genus of a knot equal the 0-shake genus of the knot?

We also find compelling the topological analogue of Conjecture 1.2, given that all known topological concordance invariants are determined by the 0-surgery of a knot.

Conjecture 1.14. *If K and K' have $S_0^3(K) \cong S_0^3(K')$ then, up to reversing the orientation of either knot, K and K' are topologically concordant.*

This paper is organized as follows. In Section 3 we introduce dualizable patterns and use them to construct pairs of knots with homeomorphic 0-surgeries. We also introduce the examples we will use to prove Theorem A. In Section 4 we show that the pairs from Section 3 have diffeomorphic 0-traces and we prove Theorem B. In Section 5 we use the d-invariants of the double branched covers of the examples from Section 3 to prove Theorem A. We conclude with a proof of Theorem C.

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3. A CONSTRUCTION OF KNOTS WITH DIFFEOMORPHIC 0-SURGERIES

3.1. Definitions and notation. Unless specifically mentioned, all maps and concordances are smooth and all knots and manifolds are both smooth and oriented. We use \cong to denote diffeomorphism of manifolds, \simeq to denote ambient isotopy of knots and links in 3-manifolds, and \sim to denote concordance of knots. All (co)homology groups are by default taken with integer coefficients. For N a properly embedded smooth submanifold of M we use $\nu(N)$ to denote a tubular neighborhood of N .

For K a knot in S^3 and p/q in the extended rationals we define $S_{p/q}^3(K)$ to be the 3-manifold obtained by p/q Dehn surgery on S^3 along K . We will use $\Sigma_n(K)$ to denote the n -fold cyclic cover of S^3 branched over K . For $n \in \mathbb{Z}$ we define the n -trace of K to be the 4-manifold obtained by attaching an n -framed two handle to the four ball along K , and refer to it as $X_n(K)$.

Let $P : S^1 \rightarrow V$ be an oriented knot in a standard solid torus $V := S^1 \times D^2$. Assume that P is not null-homologous. By the usual abuse of notation, we use P to refer to this map and its image. Define $\lambda_V = S^1 \times \{x_0\}$ for some $x_0 \in \partial D^2$ oriented so that P is homologous to a positive multiple n of λ_V . We define n to be the *(algebraic) winding number of P* . Define μ_P to be a meridian for P oriented such that the linking number of P and μ_P is 1 and define $\mu_V = \{x_1\} \times \partial D^2$ for some $x_1 \in S^1$ oriented so that μ_V is homologous to a positive multiple of μ_P . Finally define the longitude λ_P of P to be the unique framing curve of P in V which is homologous to a positive multiple of λ_V in $V \setminus \nu(P)$. Define the *geometric winding number of P* to be the minimal number of intersections of P with the meridional disk for V over all patterns in the isotopy class of P .

For any knot K in S^3 there is a canonical embedding $i_K : V \rightarrow S^3$ given by identifying V with $\overline{\nu(K)}$ such that λ_V is sent to the null-homologous curve on $\partial \nu(K)$. Then $i_K \circ P : S^1 \rightarrow S^3$ specifies the oriented knot $P(K)$ in S^3 , the *satellite of K by P* .

Let $\tau_n : S^1 \times D^2 \rightarrow S^1 \times D^2$ be the n -fold Dehn twist about a positive meridian of $S^1 \times D^2$. Then for a pattern P we define the n -twist of P to be $\tau_n(P) := \tau_n \circ P$.

Definition 3.1. A pattern P is *dualizable* if there exists $P^* : S^1 \rightarrow S^1 \times D^2 =: V^*$ such that there is an orientation preserving homeomorphism $f : V \setminus \nu(P) \rightarrow V^* \setminus \nu(P^*)$ with $f(\lambda_V) = \lambda_{P^*}$, $f(\lambda_P) = \lambda_{V^*}$, $f(\mu_V) = -\mu_{P^*}$, and $f(\mu_P) = -\mu_{V^*}$. We call P^* the *dual* of P .

Remark 3.2. There is some redundancy in this definition: if one assumes $f(\lambda_P) = \lambda_{V^*}$ and $f(\mu_V) = -\mu_{P^*}$ it is not hard to show that one can modify f in a small neighborhood of the boundary of $V \setminus \nu(P)$ so that $f(\lambda_V) = \lambda_{P^*}$ and $f(\mu_P) = -\mu_{V^*}$. Thus when checking that a pattern is dualizable or computing a dual it suffices to check $f(\lambda_P) = \lambda_{V^*}$ and $f(\mu_V) = -\mu_{P^*}$. We include all four conditions in the definition since we will often appeal to them all when discussing properties of dualizable patterns.

We also have the following idea of mirror-reversal for patterns.

Definition 3.3. Given a pattern $P : S^1 \rightarrow V$, define \overline{P} to be the pattern obtained from P by reversing the orientation of both S^1 and V ; note that \overline{P} has diagram obtained from a diagram of P by changing all crossings and the orientation of P .

Note that for P dualizable, $(\overline{P})^* = \overline{P^*}$. We warn the reader that our conventions differ from those in [GM95], in whose notation our $\overline{P^*}$ is the dual of P .

Definition 3.4. Define $\Gamma : S^1 \times D^2 \rightarrow S^1 \times S^2$ as $\Gamma(t, d) = (t, \gamma(d))$, where $\gamma : D^2 \rightarrow S^2$ is an arbitrary orientation preserving embedding. Then for any curve $\alpha : S^1 \rightarrow S^1 \times D^2$, we can define a knot in $S^1 \times S^2$ by $\widehat{\alpha} := \Gamma \circ \alpha : S^1 \rightarrow S^1 \times S^2$.

3.2. Dualizable patterns. We now describe a method of producing a large class of dualizable patterns and their duals by considering the natural inclusion of $S^1 \times D^2$ into $S^1 \times S^2$ as follows.

Proposition 3.5. A pattern P in a solid torus V is dualizable if and only if \widehat{P} is isotopic to $\widehat{\lambda_V} \simeq S^1 \times \{\Gamma(x_0)\}$ in $S^1 \times S^2$ for $x_0 \in \partial D^2$.

Proof. For the if direction, let $V^* = (S^1 \times S^2) \setminus \nu(\widehat{P})$. Since \widehat{P} is isotopic to $\widehat{\lambda_V} \simeq S^1 \times \{x_0\}$ we know V^* is a solid torus. Thus there is an identification of V^* with $S^1 \times D^2$ such that $\widehat{\lambda_P}$ is identified with $S^1 \times \{pt\} =: \lambda_{V^*}$. Let $Q = \widehat{\lambda_V} \subseteq V^*$. Let $Z = (S^1 \times S^2) \setminus \nu(\widehat{P} \sqcup Q)$, and observe that $V \setminus \nu(P) = Z = V^* \setminus \nu(Q)$. Tracing through these identifications, we see that $\mu_V \leftrightarrow -\mu_Q$ and $\lambda_P \leftrightarrow \lambda_{V^*}$. Then by Remark 3.2 P is dualizable with $P^* = Q$.

For the only if direction, observe that $S^1 \times S^2 \setminus \nu(\widehat{P})$ is diffeomorphic to the result of Dehn filling $V \setminus \nu(P)$ along μ_V . Since P is dualizable, this is diffeomorphic to the result of Dehn filling $V^* \setminus \nu(\widehat{P}^*)$ along $-\mu_{P^*}$, which is a solid torus. So \widehat{P} is a knot in $S^1 \times S^2$ with solid torus complement. By Waldhausen¹ [Wal68] all genus one Heegaard splittings of $S^1 \times S^2$ are isotopic, so we see that \widehat{P} is isotopic to $\pm \widehat{\lambda}_V$. Since by definition \widehat{P} is homotopic to a positive multiple of λ_V , \widehat{P} must be isotopic to $+\widehat{\lambda}_V$. \square

Note that Proposition 3.5 implies that every geometric winding number one pattern P is dualizable with $P^* = P$. It is also now straightforward to see that all dualizable patterns have (algebraic) winding number 1, and as a corollary of Proposition 3.5 we obtain the following fundamental group characterization of dualizability. (See Theorem 3.4 of [DR16] for a similar result.)

Corollary 3.6. *A winding number 1 pattern P in a solid torus V is dualizable if and only if $\mu_P \in \langle \langle \mu_V \rangle \rangle$, the subgroup of $\pi_1(V \setminus P)$ normally generated by μ_V .*

Proof. First, suppose that $\mu_P \in \langle \langle \mu_V \rangle \rangle$, and we will show that \widehat{P} is isotopic to $\widehat{\lambda}_V$ in $S^1 \times S^2$. Let $X = (S^1 \times S^2) \setminus \nu(\widehat{P})$. Note that $\pi_1(X) = \pi_1(V \setminus P) / \langle \langle \mu_V \rangle \rangle$ is a quotient of $\pi_1(V \setminus \nu(P)) / \langle \langle \mu_P \rangle \rangle = \pi_1(V) = \mathbb{Z}$. Since P is winding number one, we have that P is homologous to λ_V , and hence that \widehat{P} is homologous to $\widehat{\lambda}_V$. It follows that $H_1(X) \cong H_1((S^1 \times S^2) - \nu(\widehat{\lambda}_V)) \cong \mathbb{Z}$, and so we must have $\pi_1(X) \cong \mathbb{Z}$ as well. Since X has no S^2 boundary components, it follows from [Hem04] that X is homeomorphic to a solid torus. Now the extension of [Wal68] discussed in the proof of Proposition 3.5 shows that \widehat{P} must be isotopic to $\widehat{\lambda}_V$ in $S^1 \times S^2$, and hence P is dualizable.

Now suppose that P is dualizable. It suffices to show that $\mu_{V^*} \in \langle \langle \mu_{P^*} \rangle \rangle$. However, $\pi_1(V^* \setminus \nu(P^*)) / \langle \langle \mu_{P^*} \rangle \rangle \cong \pi_1(V^*)$. But certainly $[\mu_{V^*}] = 0$ in $\pi_1(V^*)$, and so we have the desired result. \square

The following example, originally due to [Bra80] and discussed in [GM95] illustrates a method for finding the dual of a dualizable pattern (see also [DR16]).

Example 3.7. The left of Figure 1 shows a pattern J in a solid torus V . We will show that J is dualizable with dual $J^* \simeq \tau_{-4}(J)$. Note that λ_V is the curve on ∂V which bounds a disk in the diagram and we also show $\widehat{J} \subset S^1 \times S^2$ (center). (We draw $S^1 \times S^2$ as $I \times S^2$ and imagine identifying the points $\partial I \times \{p\}$ for all $p \in S^2$). By isotoping the strand of \widehat{J} which runs ‘under’ for three consecutive crossings around the S^2 factor one obtains a diagram of \widehat{J} with those three crossings changed (right). From here it is straightforward to observe that \widehat{J} is isotopic in $S^1 \times S^2$ to $S^1 \times \{p_0\}$ and so J is dualizable. We now show that $J^* \simeq \tau_{-4}(J)$. The top left diagram of Figure 2 shows the link $L := \widehat{J} \sqcup \widehat{\lambda}_V \subset S^1 \times S^2$, where the link is comprised of the red and blue curves. To keep the diagram uncluttered we do not explicitly depict the ‘outer’ S^2 in our depiction of $S^1 \times S^2$ as a quotient of $I \times S^2$. We also keep track of $\widehat{\lambda}_J$ in green, so we can verify that the conditions of Definition 3.1 are satisfied. We isotope so that \widehat{J} goes to $S^1 \times \{p_0\}$ as in the bottom left of Figure 2. We call this bottom left link diagram L' .

Now let $\delta_t : S^2 \rightarrow S^2$ be the map which rotates S^2 by $2\pi t$ about an axis through the east and west poles, and define $\Delta : S^1 \times S^2 \rightarrow S^1 \times S^2$ by $\Delta(t, d) = (t, \delta_t(d))$ for all $t \in S^1$,

¹In fact, Waldhausen only proves uniqueness up to diffeomorphism. To prove uniqueness up to isotopy requires more work, which is done explicitly in, for example, Carvalho and Oertel [CO05]. For discussion of the history, see [MSZ16] and [Sch07].

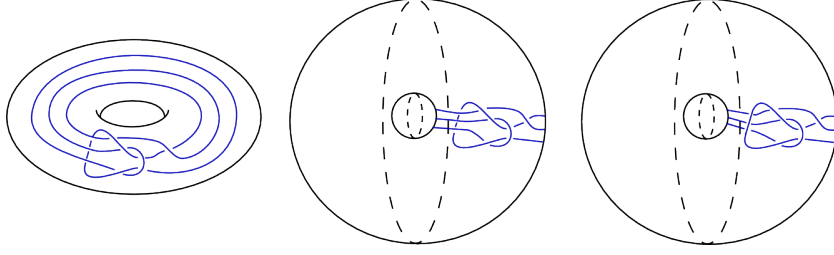


FIGURE 1. On the left, a pattern J in V the standard solid torus in S^3 . The right two diagrams show that \hat{J} is isotopic to $S^1 \times \{p_o\}$ in $S^1 \times S^2$

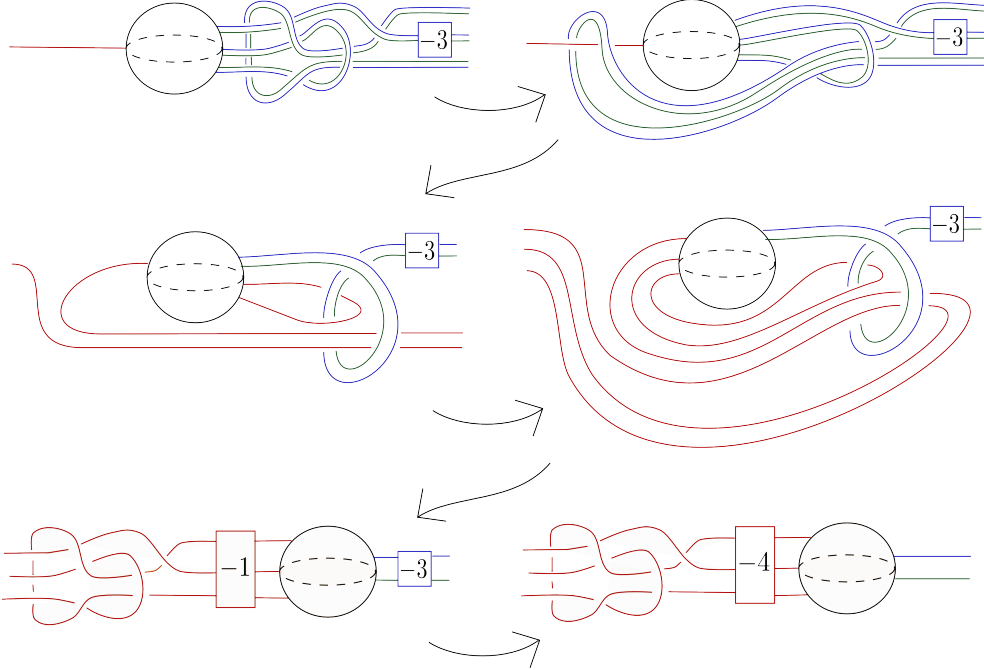


FIGURE 2. Finding $J^* \simeq \tau_{-4}(J)$ by working in $S^1 \times S^2$.

$d \in S^2$. Observe that Δ^3 is a self-homeomorphism of $S^1 \times S^2$ which takes L' to the link $L'' = (S^1 \times \{x_0\}) \sqcup \eta$ shown in the bottom right of Figure 2.

The composition of Δ^3 with the homeomorphism induced by the isotopy of L to L' gives a self-homeomorphism of $S^1 \times S^2$ which takes L to L'' , and hence a homeomorphism $f : (S^1 \times S^2) \setminus \nu(L) \rightarrow (S^1 \times S^2) \setminus \nu(L'')$. Note that $(S^1 \times S^2) \setminus \nu(L) = V \setminus \nu(J)$. Identify $(S^1 \times S^2) \setminus (S^1 \times \{x_0\})$ as a solid torus V^* , where the identification is made so that $f(\lambda_J) = \lambda_{V^*}$. Finally, we can identify $(S^1 \times S^2) \setminus \nu(L'')$ as $V^* \setminus \nu(J^*)$ for some $J^* \hookrightarrow V^*$ so that f gives a homeomorphism from $V \setminus \nu(J)$ to $V^* \setminus \nu(J^*)$ with the hypotheses of Definition 3.1 satisfied.

Theorem 3.8. [Bra80] *If P is a dualizable pattern with dual P^* , then there is a homeomorphism $\phi : S_0^3(P(U)) \rightarrow S_0^3(P^*(U))$.*

Proof. Construct $S_0^3(P(U))$ by Dehn filling $V \setminus \nu(P)$ along λ_P and λ_V . Since P is dualizable, this is diffeomorphic to $V^* \setminus \nu(P^*)$ Dehn filled along λ_{V^*} and λ_{P^*} , which is $S_0^3(P^*(U))$. \square

Proposition 3.9. *If P is dualizable then so is $\tau_n(P) = \tau_n \circ P$, with dual pattern $(\tau_n P)^* = \tau_{-n}(P^*)$.*

Proof. We construct $(\tau_n P)^*$. For this argument we will draw P as the closure in $S^1 \times D^2$ of a $2r + 1$ strand tangle which we will denote with a box labeled P .

In Figure 3 we start to construct $(\tau_n P)^*$ as in Example 3.7. In order to construct a homeomorphism between $V \setminus \nu(P)$ and the complement of some pattern in a solid torus V^* we look for a self-homeomorphism of $S^1 \times S^2$ which takes $\widehat{\tau_n(P)} \sqcup \widehat{\lambda_V}$ to a link L' such that $\widehat{\tau_n(P)}$ is sent to an $S^1 \times \{pt\}$ component of L' . We also keep track of a copy of $\Gamma(\lambda_{\tau_n(P)})$ in green; note that within the tangle-box the green curve is λ_P framing the blue curve P .

As a first step we apply Δ^{-n} to remove the n twists from $\widehat{\tau_n(P)}$. Then the resulting link is $\widehat{P} \sqcup \widehat{\lambda_V}$, and the green curve is a copy of $\Gamma(\lambda_P - n\mu_P)$. Then since P is dualizable there is a homeomorphism to the third diagram in Figure 3. Applying Δ^n we get a link L' in which the images of $\widehat{\tau_n(P)}$ and $\Gamma(\lambda_{\tau_n(P)})$ are unlinked and isotopic to $S^1 \times \{p_0\}$. From this diagram we can read off $(\tau_n P)^*$ and we see that it is $\tau_{-n}(P^*)$. \square

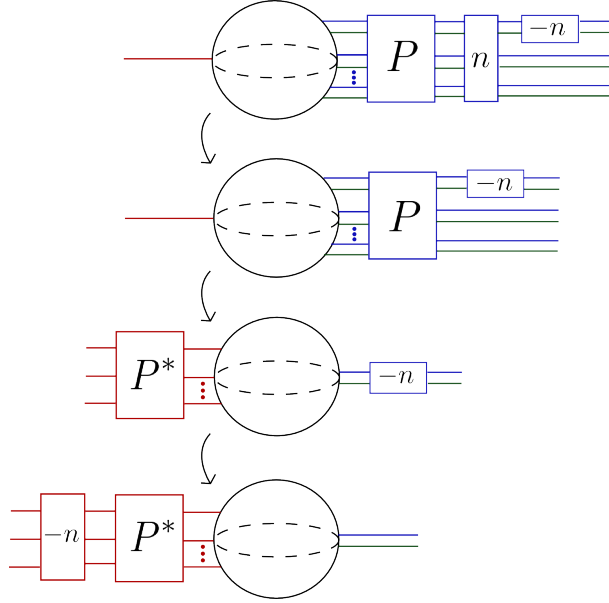


FIGURE 3. The dual of $\tau_n P$ is $\tau_{-n} P^*$.

Proposition 3.10. *Let P be a dualizable pattern with dual P^* . Then for any $n \in \mathbb{Z}$, $S_n^3(P(U)) \cong S_n^3[(\tau_n P^*)(U)]$.*

Proof. It is straightforward to generalize the proof of Theorem 3.8 to this setting. \square

4. EXTENDING THE 0-SURGERY DIFFEOMORPHISM ACROSS THE 0-TRACE

Our goal of this section will be to prove the following theorem.

Theorem 4.1. *Let P be a dualizable pattern. Then there is a diffeomorphism $\Phi : X_0(P(U)) \rightarrow X_0(P^*(U))$.*

First, we recall a result of Akbulut which first appeared in [Akb77]. We refer the reader to proofs in [AJOT13] and [Akb16]. For the details of handle calculus see [GS99].

Lemma 4.2. *Let M and N be four-manifolds with a homeomorphism $\psi : \partial M \rightarrow \partial N$. If the following are true of ψ , then there exists a diffeomorphism $\Psi : M \rightarrow N$ such that $\Psi|_{\partial} = \psi$.*

- (1) *There exists some $K : S^1 \rightarrow \partial M$ which bounds a smoothly embedded disk D_K in M and with the property that $\psi(K)$ bounds a smoothly embedded disk $D_{\psi(K)}$ in N .*
- (2) *Let D'_K be a section of $\nu(\overline{D_K})$ and $D'_{\psi(K)}$ be a section of $\nu(\overline{D_{\psi(K)}})$. Then $\psi(\partial D'_K)$ and $\partial D'_{\psi(K)}$ induce the same framing on $\psi(K)$.*
- (3) *$M \setminus \nu(D_K) \cong N \setminus \nu(D_{\psi(K)}) \cong B^4$.*

Proof of Theorem 4.1. We will check that for $M := X_0(P(U))$ and $N := X_0(P^*(U))$ the homeomorphism $\phi : \partial M \rightarrow \partial N$ as in Theorem 3.8 satisfies the conditions of Lemma 4.2.

Let K be $\mu_{P(U)} := i_U(\mu_P)$ in ∂M . Then K bounds a disk D_K in M , namely a disk in S^3 with its interior pushed slightly into $B^4 \subseteq X_0(P(U)) = M$. One checks that $\partial D'_K$ induces the 0-framing on K . By inspection of ϕ we see that $\phi(K)$ is $i_U(\mu_{V^*}) \subset \partial N$. Just as we saw for K , the knot $\phi(K)$ bounds a disk $D'_{\phi(K)}$ in N and $\partial D'_{\phi(K)}$ induces the 0-framing on $\phi(K)$. Further inspection of ϕ shows that $\phi(\partial D'_K)$ induces the 0-framing on $\phi(K)$. So conditions (1) and (2) are satisfied.

Figure 4 demonstrates handle diagrams of $M \setminus \nu(D_K)$ and $N \setminus \nu(D_{\phi(K)})$. It is clear that $M \setminus \nu(D_K) \cong B^4$.

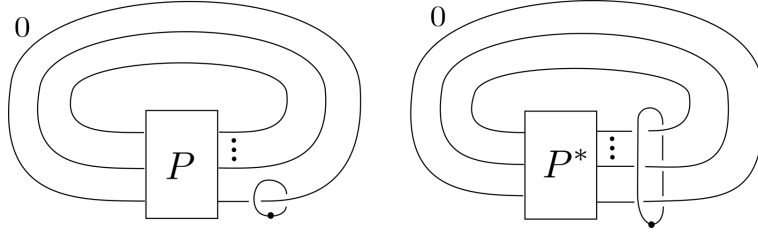


FIGURE 4. Handle diagrams for $M \setminus \nu(D_K)$ and $N \setminus \nu(D_{\phi(K)})$, respectively

Observe that $\partial(N \setminus \nu(D_{\phi(K)}))$ can be interpreted as a Dehn surgery on $\widehat{P^*} \subset S^1 \times S^2$. By Proposition 3.5, there is some isotopy of $\widehat{P^*}$ to $S^1 \times \{pt\}$. This implies that in the given handle decomposition of $N \setminus \nu(D_{\phi(K)})$ there is some sequence of slides of the two handle over the one handle which results in a handle diagram for $N \setminus \nu(D_{\phi(K)})$ as in Figure 5. It is clear from Figure 5 that $N \setminus \nu(D_{\phi(K)}) \cong B^4$.

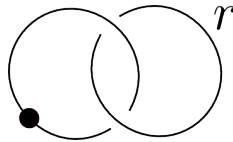


FIGURE 5. A handle diagram for $N \setminus \nu(D_{\phi(K)})$, where r is some integer

□

For a pattern Q in a solid torus V , let $j_Q : S^1 \times D^2 \rightarrow V$ be an identification of $S^1 \times D^2$ with $\overline{\nu(Q)}$ such that $j_Q(S^1 \times \{pt\}) = \lambda_Q$. For P a pattern in $S^1 \times D^2$ we define $P \circ Q := j_Q \circ P : S^1 \rightarrow V$.

Proposition 4.3. *Let P, Q be dualizable patterns. Then $P \circ Q$ is dualizable with dual $Q^* \circ P^*$.*

Proof. For clarity, we write each pattern $R \in \{P, Q, P \circ Q, Q^* \circ P^*\}$ as living in a solid torus V_R . Observe that $V_{P \circ Q} \setminus \nu(P \circ Q)$ can be written as the union of $(V_Q \setminus \nu(Q))$ with $(V_P \setminus \nu(P))$ via an identification of λ_{V_P} with λ_Q and μ_{V_P} with μ_Q . Since P and Q are dualizable, we see that $V_{P \circ Q} \setminus \nu(P \circ Q) \cong (V_Q^* \setminus \nu(Q^*)) \cup (V_P^* \setminus \nu(P^*))$ glued together so that λ_{P^*} is identified with $\lambda_{V_Q^*}$ and μ_{P^*} is identified with $-\mu_{V_Q^*}$. But this gives a description of $V_{Q^* \circ P^*} \setminus \nu(Q^* \circ P^*)$ in which it is straightforward to check that the longitude and meridian conditions of Definition 3.1 are satisfied. \square

Corollary 4.4. *For P a dualizable pattern and K a knot in S^3 , $X_0(P(K)) \cong X_0(P^*(U) \# K)$.*

Proof. Let $K_{\#}$ be the geometric winding number one pattern with $K_{\#}(U) \simeq K$. Observe that $K_{\#}$ is dualizable and self dual. Then consider $(P \circ K_{\#})(U)$ and apply Proposition 4.3 and Theorem 4.1. \square

Corollary 4.5. *For P dualizable, $\overline{P^*}(P(U)) \sim U \sim P(\overline{P^*}(U))$*

Proof. Let $P_{\#}$ be the geometric winding number one pattern with $P_{\#}(U) = P(U)$. Then by Theorem 4.1 and Proposition 4.3 we have

$$X_0(\overline{P^*}(P(U))) \cong X_0(\overline{P^*}(P_{\#}(U))) \cong X_0((\overline{P^*} \circ P_{\#})(U)) \cong X_0((P_{\#} \circ \overline{P})(U)) \cong X_0(P(U) \# \overline{P}(U))$$

It follows from Definition 3.3 that $P(U) \# \overline{P}(U)$ is slice, so by Corollary 1.8, we have that $\overline{P^*}(P(U))$ is slice as well. The other concordance follows similarly. \square

Proof of Theorem B. Let $K_{\#}$ be the geometric winding number one pattern with $K_{\#}(U) = K$ as above. Observe that

$$\overline{P^*}(P(K)) \# -K = (\overline{K_{\#}} \circ \overline{P^*})(P \circ K_{\#})(U) = (\overline{K_{\#}} \circ \overline{P^*})(P \circ K_{\#})(U).$$

By Proposition 4.3, we have that $(K_{\#} \circ P^*)$ is the dual of $(P \circ K_{\#})$. Thus by Corollary 4.5 $\overline{P^*}(P(K)) \# -K$ is slice and $\overline{P^*}(P(K)) \sim K$. The argument that $P(\overline{P^*}(K)) \sim K$ is analogous. \square

The following generalization of Theorem 4.1 is proved via an identical argument; we leave the proof to the reader.

Theorem 4.6. *For P a dualizable pattern, $X_n(P(U)) \cong X_n[(\tau_n P^*)(U)]$.*

5. DISTINGUISHING CERTAIN $P(U)$ AND $P^*(U)$ IN CONCORDANCE

Now we consider the double branched covers of knots $K_k := (\tau_{2k-1}J)(U)$ for $k \in \mathbb{Z}$ and J the dualizable pattern of Example 3.7. A standard argument shows that if two knots K and K' are concordant, then their double branched covers $\Sigma_2(K)$ and $\Sigma_2(K')$ are rational homology cobordant. We will use the d-invariants of Ozsváth and Szabó to show that there is no such cobordism between the double branched covers of certain K_k and K_j . For the reader's convenience, we state the facts we will need about the behavior of d-invariants.

Theorem 5.1 (Theorem 9.6 of [OS03a]). *To a rational homology sphere Y^3 and a Spin^c -structure \mathfrak{s} there is an invariant $d(Y, \mathfrak{s}) \in \mathbb{Q}$ satisfying the following properties: If W is a cobordism from Y_0 to Y_1 , then for any $\mathfrak{s} \in \text{Spin}^c(W)$*

- (1) $d(Y_1, \mathfrak{s}|_{Y_1}) \geq d(Y_0, \mathfrak{s}|_{Y_0}) + \frac{c_1(\mathfrak{s})^2 + \beta_2(W)}{4}$ if W is negative definite.
- (2) $d(Y_1, \mathfrak{s}|_{Y_1}) \leq d(Y_0, \mathfrak{s}|_{Y_0}) + \frac{c_1(\mathfrak{s})^2 - \beta_2(W)}{4}$ if W is positive definite.
- (3) $d(Y_1, \mathfrak{s}|_{Y_1}) = d(Y_0, \mathfrak{s}|_{Y_0})$ if W is a rational homology cobordism.

Note that $\text{Spin}^c(Y)$ can be non-canonically put in bijective correspondence with $H^2(Y)$ and so an integer homology sphere Y has a single Spin^c -structure and hence a single d -invariant, which we refer to as $d(Y)$.

Proposition 5.2 (Corollary 1.5 of [OS03b]). *Let K be an alternating knot. Then*

$$d(S_1^3(K)) = 2 \min \left\{ 0, - \left\lceil \frac{-\sigma(K)}{4} \right\rceil \right\}$$

First observe that K_k , shown on the left of Figure 6, has a surgery description as on the right of Figure 6. Let η be the red $(-\frac{1}{2k})$ -framed curve and γ be the blue $(+1)$ -framed curve. Some isotoping gives us the surgery description on the left of Figure 7. From this we obtain the surgery description for $Y_k := \Sigma_2(K_k)$ on the right of Figure 7. For the reader's convenience we say a few words justifying the surgery coefficients, where for any curve σ in a diagram we use λ_σ^{bb} to refer to the blackboard-framed longitude of σ . We see on the left

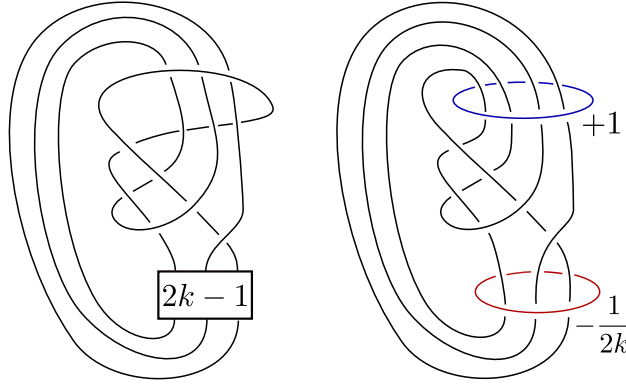


FIGURE 6. K_k (left) has an unknotted surgery description (right).

side of Figure 7 that η has writhe 0, so we can write $\text{fr}_\eta = -\mu_\eta + 2k\lambda_\eta^{bb}$. The preimage of μ_η under the branched covering map is $\mu_{\tilde{\eta}}$ and the preimage of $2\lambda_\eta^{bb}$ is $\lambda_{\tilde{\eta}}^{bb}$. Since $\tilde{\eta}$ has writhe 0 the preimage of fr_η is

$$\text{fr}_{\tilde{\eta}} = -\mu_{\tilde{\eta}} + k\lambda_{\tilde{\eta}}^{bb} = -\mu_{\tilde{\eta}} + k\lambda_{\tilde{\eta}}$$

so we have surgery coefficient of $-1/k$ for $\tilde{\eta}$. Similarly, on the left of Figure 7 we see that γ has writhe $+2$. So $\text{fr}_\gamma = \mu_\gamma + \lambda_\gamma = -\mu_\gamma + \lambda_\gamma^{bb}$. Since γ lifts to two closed curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, the preimage of μ_γ is $\mu_{\tilde{\gamma}_i}$ and of λ_γ^{bb} is $\lambda_{\tilde{\gamma}_i}^{bb}$ for $i = 1, 2$. Since the $\tilde{\gamma}_i$ both have writhe $+2$ we have

$$\text{fr}_{\tilde{\gamma}_i} = -\mu_{\tilde{\gamma}_i} + \lambda_{\tilde{\gamma}_i}^{bb} = -\mu_{\tilde{\gamma}_i} + (\lambda_{\tilde{\gamma}_i} + 2\mu_{\tilde{\gamma}_i}) = \mu_{\tilde{\gamma}_i} + \lambda_{\tilde{\gamma}_i} \text{ for } i = 1, 2$$

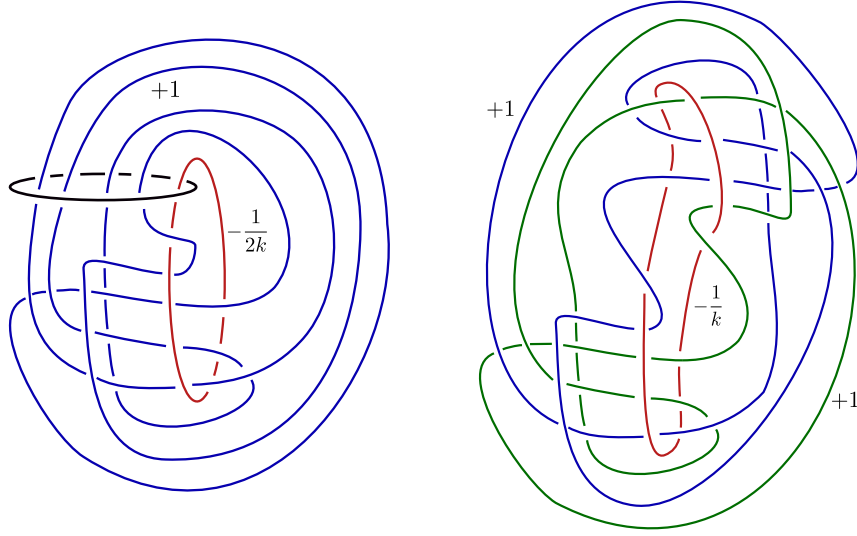


FIGURE 7. K_k now appears as the standard unknot (left), which gives a surgery description for $Y_k = \Sigma_2(K_k)$ (right)

so we have surgery coefficients of $+1$ for the $\tilde{\gamma}_i$.

Proposition 5.3. *Let K_k and Y_k be as above. Then Y_k is an integer homology sphere and hence has a single d -invariant $d(Y_k)$ satisfying*

$$d(Y_{-k}) \leq d(Y_0) \leq d(Y_k) \text{ for } k \in \mathbb{N}.$$

Proof. From Figure 7, we see that Y_0 has a surgery description $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ whose linking matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and so Y_0 is a homology sphere. Also, Y_k is obtained from Y_0 by $(-1/k)$ -surgery on $\tilde{\eta}$, which has zero algebraic linking with γ_1 and γ_2 , and so Y_k is also a homology sphere. Now recall that $k > 0$, and observe that $-1/k$ has continued fraction expansion $[-1, -2, -2, \dots, -2]$, where there are $(k-1)$ occurrences of -2 . Therefore we have a surgery diagram for Y_k that differs from that of Figure 7 only in a small neighborhood of a point on $\tilde{\eta}$ as indicated in Figure 8 (see [GS99] for justification). Let W_k be the four-manifold

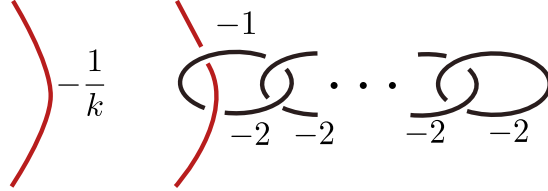


FIGURE 8. An integer surgery diagram for Y_k .

obtained from $Y_0 \times I$ by attaching k 2-handles, one with (-1) - framing and $(k-1)$ with (-2) -framing as indicated in Figure 8. Notice that W_k is a cobordism from Y_0 to Y_k and by

standard calculations

$$H^j(W_k) \cong H_j(W_k) \cong \begin{cases} 0 & j = 1 \\ \mathbb{Z}^k & j = 2 \\ \mathbb{Z} & j = 3 \end{cases}.$$

Let S_i be the union of the core of the i th 2-handle of W_k with a null-homology of the attaching circle of this handle in Y_0 . Notice that S_1, \dots, S_k generate $H_2(W_k)$. With respect to this basis the intersection form of W_k is given by the $(k \times k)$ linking matrix of the attaching circles, which is

$$Q_k = \begin{bmatrix} -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & -2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & -2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & -2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & -2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & -2 \end{bmatrix}.$$

It is straightforward to show inductively that given $v = [v_1 \dots v_k]$ we have

$$vQ_kv^T = -(v_1 + v_2)^2 - (v_2 + v_3)^2 - \dots - (v_{k-1} + v_k)^2 - v_k^2.$$

This gives an isomorphism from Q_k to the standard negative-definite intersection form on \mathbb{Z}^k . If \mathfrak{s} is any Spin^c structure on W_k , then \mathfrak{s} must restrict on ∂W_k to the unique Spin^c structures on Y_0 and Y_k , and Theorem 9.6 of [OS03a] tells us that

$$d(Y_k) \geq d(Y_0) + \frac{c_1(\mathfrak{s})^2 + \beta_2(W_k)}{4}.$$

The map $c_1 : \text{Spin}^c(W_k) \rightarrow H^2(W_k)$ has image consisting exactly of characteristic vectors; i.e. v with $v \cdot w \equiv w \cdot w \pmod{2}$ for all $w \in H^2(W_k)$. So in order to finish the proof it is enough to show that there is some characteristic vector $\xi \in H^2(W_k) \cong \mathbb{Z}^k$ with $\xi \cdot \xi + k \geq 0$. But this follows immediately from our isomorphism to the standard negative-definite intersection form, since with respect to the basis for $H_2(W_k)$ given by $\{v_1 + v_2, v_2 + v_3, \dots, v_{k-1} + v_k, v_k\}$ the vector $\xi = [1, 1, \dots, 1, 1]$ is easily seen to be characteristic and satisfy $\xi \cdot \xi = -k$.

The argument to show $d(Y_{-k}) \leq d(Y_0)$ is almost identical: observe that Y_{-k} is obtained from Y_0 by $1/k$ surgery on $\tilde{\eta}$, note that $1/k$ has continued fraction expansion $[1, 2, \dots, 2]$ (with $k-1$ occurrences of ‘2’), build a cobordism W_k^+ from Y_0 to Y_k with k 2-handles, note that W_k^+ is positive-definite, and apply Theorem 9.6 of [OS03a]. \square

In fact, the proof of Proposition 5.3 extends to show the following.

Proposition 5.4. *Let L be a knot in S^3 and η be an unknot which has linking number ± 1 with L . Suppose that $\Sigma_n(L)$ is an integer homology sphere for some $n \in \mathbb{N}$. For $k \in \mathbb{Z}$, let $\tau_{nk}(L, \eta)$ denote the nk -fold Rolfsen twist of L along η . Then for any $k \in \mathbb{N}$,*

$$d(\Sigma_n(\tau_{-nk}(L, \eta))) \leq d(\Sigma_n(L)) \leq d(\Sigma_n(\tau_{nk}(L, \eta)))$$

Proof. The argument goes as follows: unknot L by (± 1) -surgery on unknotted curves $\{\gamma_i\}$ which algebraically link L and η 0 times to construct a surgery diagram for $\tau_{nk}(L, \eta)$. (One can always do this by choosing small surgery curves about unknotting crossings for L .) Adding the curve η with framing $(-1/nk)$ gives a surgery diagram for $\tau_{nk}(L, \eta)$. Then follow the above algorithm to obtain a surgery diagram for $\Sigma_n(\tau_{nk}(L, \eta))$. It suffices to show that η lifts to a single null-homologous curve with framing $(-1/k)$ in the surgery diagram for $\Sigma_n(L)$ in order to proceed exactly as in the proof of Proposition 5.3.

One checks that η lifts to a curve with framing $-1/k$ exactly as we lifted coefficients above. Since η has linking number one with L , it has preimage a single curve $\tilde{\eta}$ in our surgery diagram for $\Sigma_n(\tau_{nk}L)$. Since η has zero linking with each of the γ_i we have that $\tilde{\eta}$ has zero linking with each of the n lifts of each γ_i , hence $\tilde{\eta}$ is null-homologous. \square

Proposition 5.4 implies that the sequence $\{d(\Sigma_2((\tau_{2k-1}J)(U)))\}$ is nondecreasing in k . We also observe that Proposition 5.4 is not hard to extend to the case when $\Sigma_n(K)$ is not an integer homology sphere, but since it is not needed for our purposes we do not do so here.

Example 5.5. Observe that when $k = 0$, $\tilde{\eta}$ has surgery coefficient ∞ , and so we can delete it from the diagram of Figure 7 and with a bit of isotoping obtain the left side of Figure 9. Blowing down $\tilde{\gamma}_2$ gives the surgery diagram on the right side of Figure 9. So $Y_0 = S_1^3(5_2)$ and Corollary 1.5 of [OS03b] tells us that

$$(4) \quad d(\Sigma((\tau_{-1}P)(U))) = d(S_1^3(5_2)) = 2 \min \left\{ 0, - \left\lceil \frac{-\sigma(5_2)}{4} \right\rceil \right\} = 2 \min \left\{ 0, - \left\lceil \frac{2}{4} \right\rceil \right\} = -2$$

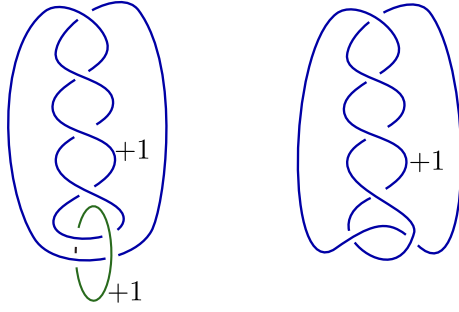


FIGURE 9. Surgery diagrams for $Y_0 = \Sigma_2((\tau_{-1}P)(U))$

Example 5.6. We use a new surgery diagram for K_1 to compute $d(Y_1)$. The left side of Figure 10 depicts K_1 in the standard S^3 and the middle gives a surgery diagram for S^3 in which K_1 appears unknotted. An isotopy takes K to a curve with no self-crossings as on the right of Figure 10. We therefore have a surgery diagram for Y_1 as on the left of Figure 11, which after some isotopy appears as in the center of Figure 11. Now observe that the 4-manifold obtained by adding one 0-framed 2-handle to $S^1 \times B^3$ as on the right of Figure 11 has $\partial Z = \Sigma(K_1)$. Since Z consists of a 0-handle and an algebraically canceling 1- and 2-handle pair, Z is an integer homology ball. It follows from Theorem 1.2 of [OS03a] that $d(\Sigma(K_1)) = 0$.

Proof Of Theorem A. For any $k \in \mathbb{N}$, as before let $K_k = (\tau_{2k-1}J)(U)$. We now let $K'_k = (\tau_{-2k-3}J)(U)$. We will see in Section 6 that when $k \neq k'$ an Alexander polynomial computation shows that K_k and $K_{k'}$ are distinct knots.

By Example 3.7 and Theorem 4.1, we have that K_k and K'_k have diffeomorphic 0-traces for any $k \in \mathbb{N}$. However, by the propositions and examples of this section, we have that

$$d(\Sigma_2(K'_k)) \leq d(\Sigma_2(K_0)) = -2 < 0 = d(\Sigma_2(K_1)) \leq d(\Sigma_2(K_k)), k \in \mathbb{N}.$$

Therefore by Theorem 1.2 of [OS03a], $\Sigma_2(K'_k)$ and $\Sigma_2(K_k)$ are not smoothly rationally homology cobordant. Thus K'_k and K_k are not smoothly concordant. \square

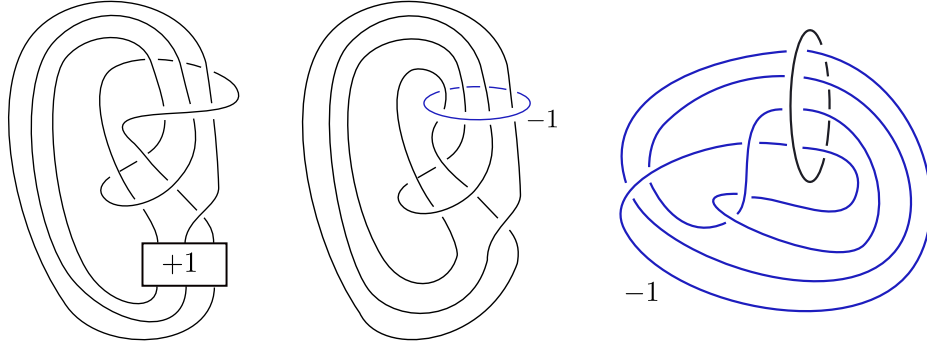


FIGURE 10. The knot $(\tau_1 P)(U)$ (left) and two surgery diagrams for it (center, right).

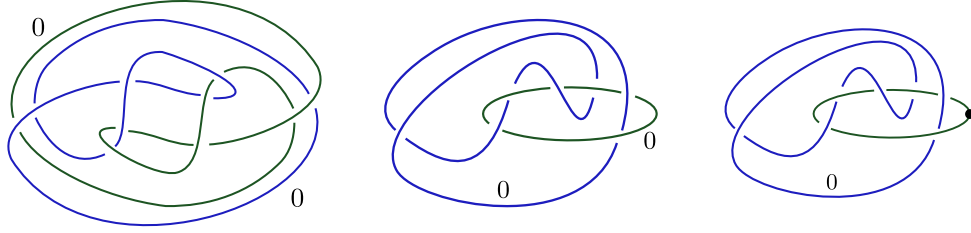


FIGURE 11. Surgery diagrams for $\Sigma_2(K_1)$ (left, center) and a Kirby diagram for Z^4 with $\partial Z = \Sigma_2(K_1)$ (right).

Proof of Theorem C. Observe that if a pattern P acts by connected sum it must act by connected sum with $P(U)$ and so $P(U)$ and $P^*(U)$ must be concordant, since

$$U \sim P(P^{-1}(U)) \sim P(U) \# P^{-1}(U) \sim P(U) \# \overline{P^*}(U).$$

But the examples in the proof of Theorem A gives us infinitely many dualizable, hence invertible, patterns for which this is not the case. \square

Remark 5.7. In fact, for any dualizable pattern P with $P(U) \not\sim P^*(U)$, we can construct a dualizable pattern Q such that $Q(U)$ is slice but the action of Q on the smooth concordance group is nontrivial as follows. Let $-P_\#$ be the geometric winding number one pattern with $-P_\#(U) = -P(U)$ and let $Q = -P_\# \circ P$. Then Q is a dualizable pattern with $Q(U) = P(U) \# -P(U) \sim U$. However,

$$Q(-P^*(U)) = P(-P^*(U)) \# -P(U) \sim -P(U) \not\sim -P^*(U).$$

Note that this stands in stark contrast to the topological setting, in which it is still unknown whether there exist *any* algebraic winding number one pattern S with $S(U)$ topologically slice and yet $S(K)$ not topologically concordant to K .

Applying Corollary 4.4 to such a pattern Q we get that for any non-trivial knot K there exists $K' = K \# Q^*(U)$ with $K \sim K'$ and a knot $K'' = Q(K) \not\sim K'$ such that $X_0(K') \cong X_0(K'')$. Note that the fact that $K \# Q^*(U) \not\sim Q(K)$ follows from the uniqueness of satellite descriptions of knots. This leads us to the following question.

Question 5.8. For which concordance classes $[K]$ does there exist $[K'] \neq [K]$ with representatives K_0 and K'_0 with diffeomorphic 0-traces?

Theorem A tells us that such concordance classes exist, and Corollary 1.8 tells us that $[K]$ cannot be trivial. Further, as noted in the introduction, for a given $[K]$ there are many restrictions on the possible choices of $[K']$. One might also ask how many distinct $[K']$ are possible: can one find infinitely many concordance classes of knots all of which have representatives sharing a 0-trace? The techniques of annulus twisting seem well-suited to create candidate knots, but finding computable invariants capable of distinguishing these knots in smooth concordance appears to be a difficult problem.

6. AN INTERESTING FAMILY OF KNOTS.

We close by considering the family of knots $\{(\tau_n J)(U)\}_{n \in \mathbb{Z}}$, where J is the pattern from Example 3.7. An explicit computation with Seifert matrices, which we omit, shows that, for $n \geq -2$,

$$\Delta_{(\tau_n J)(U)}(t) = (t-1)^2(t^{2n+4} + 1) + (2t^2 - 3t + 2)t^{n+2}$$

Since the Alexander polynomial of K is an invariant of the 0-surgery of K , Theorem 3.8 and Proposition 3.9 allow us to compute $\Delta_{(\tau_n J)(U)}(t)$ for all $n \in \mathbb{Z}$. This shows that $(\tau_n J)(U)$ is distinct from $(\tau_{n'} J)(U)$ for $n' \notin \{n, -4-n\}$. It certainly follows from the proof of Theorem A that this remains true when $n' = -4-n$ and n is odd. We expect that when $n' = -4-n$ and n is even we will still have $(\tau_n J)(U) \not\cong (\tau_{n'} J)(U)$ but do not pursue that here.

What makes this family remarkable is that for any pair of integers n and n' Example 3.7 and Theorem 4.6 show that there is an integer r such that $X_r((\tau_n J)(U)) \cong X_r((\tau_{n'} J)(U))$. In fact, for any n there is at most a single integer s such that $X_s((\tau_n J)(U))$ is not diffeomorphic to $X_s((\tau_m J)(U))$ for some $m \in \mathbb{Z}$.

We also point out that all the knots in this family have four genus equal to 1. To show the four genera are all one or less we point out that for any n there is a band move taking $(\tau_n J)(U)$ to a Hopf link. For n even, we can compute that the determinant of $(\tau_n J)(U)$ equals 15 and hence $(\tau_n J)(U)$ is not slice. For n odd, we note that by Theorem 4.1 and Corollary 1.8 it suffices to show that $(\tau_n J)(U)$ is not slice for $n \leq -1$. However, we will see in Proposition 5.3 and Example 5.5 that when $n \leq -1$ is odd we have

$$d(\Sigma_2((\tau_n J)(U))) \leq d(\Sigma_2((\tau_{-1} J)(U))) = -2 < 0,$$

and it follows that $(\tau_n J)(U)$ is not slice.

In fact, we have $g_4((\tau_n J)(U) \# -(\tau_{n'} J)(U)) \leq 1$ for any pair of integers n and n' , as shown by the four band moves shown in Figure 12, which take $(\tau_n J)(U) \# -(\tau_{n'} J)(U)$ to a three component unlink. (In fact, one can show that $g_4((\tau_n J)(K) \# -(\tau_{n'} J)(U) \# -K) \leq 1$ by a

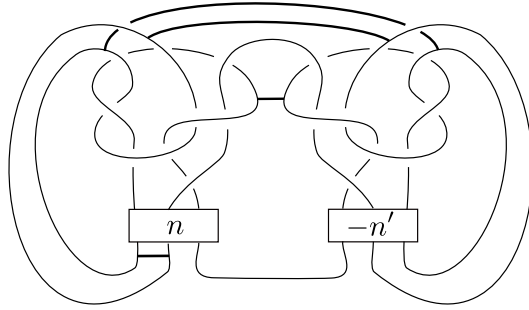


FIGURE 12. The four genus of $(\tau_n J)(U) \# -(\tau_{n'} J)(U)$ is no more than 1.

virtually identical argument.) Since for odd $n \neq n'$ we have that $(\tau_n J)(U)$ and $(\tau_{n'} J)(U)$ are not concordant, this gives an infinite family of knots any two of which are distance exactly 1 from each other under the metric $d(K, J) := g_4(K \# - J)$.

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